

# On the Densely Pythagorean Rational Approximability of Euclidean Spaces and Spheres of the Same Dimension

C.P. ANIL KUMAR

**ABSTRACT.** In this article we prove the equivalence of two conjectures namely the densely elliptic rational approximability of euclidean spaces and the densely 2-hyperbolic rational approximability of unit spheres of the same dimension in Theorem 17. Finally this article gives rise to an interesting Question 1 about the type of rational approximability of infinite geometric sets in a finite dimensional euclidean space after proving Theorems 9, 20 concerning densely rational approximability.

## 1. Introduction

We begin this introduction with a few definitions.

**DEFINITION 1.** Let  $(X, d)$  be a metric space. We say that  $X$  is a rational set if the distance set

$$\Delta(X) = \{d(x, y) \mid x, y \in X\} \subset \mathbb{Q}^+ \cup \{0\}.$$

**DEFINITION 2.** We say a rational number  $r = \frac{p}{q}, p, q \in \mathbb{Z}, \gcd(p, q) = 1$ , is an elliptic rational if  $p^2 + q^2 = \square$ . Let

$$\mathbb{Q}_{\text{elliptic}} = \{r \in \mathbb{Q} \mid r \text{ is an elliptic rational}\}.$$

Let  $k > 0$  be a positive rational. Let  $(X, d)$  be a metric space. We say that  $X$  is a  $k$ -elliptic rational set if the distance set

$$\Delta(X) = \{d(x, y) \mid x, y \in X\} \subset \mathbb{Q}^+ \cup \{0\}$$

and there exists a point  $P \in X$  such that

$$\Delta_P(X) = \{d(P, x) \mid x \in X\} \subset k(\mathbb{Q}_{\text{elliptic}}^+ \cup \{0\}).$$

Let  $(X, d)$  be a metric space. We say that  $X$  is an elliptic rational set if  $X$  is a  $k$ -elliptic rational set for some positive rational  $k$ .

---

2010 *Mathematics Subject Classification.* Mathematics Subject Classifications: 14G05, 11J17, 11J83.

*Key words and phrases.* rational set, rational distances, spheres, euclidean spaces, rational approximation.

DEFINITION 3. We say a rational number  $r = \frac{p}{q}, p, q \in \mathbb{Z}, \gcd(p, q) = 1$ , is a hyperbolic rational if  $p^2 - q^2 = \pm 1$ . Let

$$\mathbb{Q}_{\text{hyperbolic}} = \{r \in \mathbb{Q} \mid r \text{ is a hyperbolic rational}\}.$$

Let  $k > 0$  be a positive rational. Let  $(X, d)$  be a metric space. We say that  $X$  is a  $k$ -hyperbolic rational set if the distance set

$$\Delta(X) = \{d(x, y) \mid x, y \in X\} \subset \mathbb{Q}^+ \cup \{0\}$$

and there exists a point  $P \in X$  such that

$$\Delta_P(X) = \{d(P, x) \mid x \in X\} \subset k(\mathbb{Q}_{\text{hyperbolic}}^+ \cup \{0\}).$$

Let  $(X, d)$  be a metric space. We say that  $X$  is a hyperbolic rational set if  $X$  is a  $k$ -hyperbolic rational set for some positive rational  $k$ .

DEFINITION 4. Let  $(X, d)$  be a metric space. We say that  $X$  is a pythagorean rational set if  $X$  is either a hyperbolic rational set or an elliptic rational set.

DEFINITION 5. Let  $k$  be a positive integer. Let  $f \in \mathbb{Z}[X_1, X_2, \dots, X_k]$  be a polynomial with integer coefficients in  $k$  variables. Let

$$X = \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k \mid f(x_1, x_2, \dots, x_k) = 0\}$$

be an infinite set. Let  $(\mathbb{R}^k, d)$  be the metric space, where  $d$  is the usual euclidean metric.

We say the subset  $X \subset \mathbb{R}^k$  is densely rational approximable if there exists a rational set  $Y \subset X$  such that  $\overline{Y} = X$ . We say the subset  $X \subset \mathbb{R}^k$  has a dense set (topologically) of rational points if

$$\overline{X \cap \mathbb{Q}^k} = X.$$

We say  $X$  is rationally dense if there exists a rational set  $Y \subset X \cap \mathbb{Q}^k$  such that  $\overline{Y} = X$ .

We say the subset  $X \subset \mathbb{R}^k$  is densely  $l$ -pythagorean, elliptic, hyperbolic rational approximable if there exists a  $l$ -pythagorean, elliptic, hyperbolic (respectively) rational set  $Y \subset X$  such that  $\overline{Y} = X$ .

We say the subset  $X \subset \mathbb{R}^k$  is densely pythagorean, elliptic, hyperbolic rational approximable if it is densely  $k$ -pythagorean, elliptic, hyperbolic (respectively) rational approximable for some positive rational  $l$ .

We say  $X$  is pythagorean, elliptic, hyperbolic rationally dense if there exists a pythagorean, elliptic, hyperbolic rational set  $Y \subset X \cap \mathbb{Q}^k$  such that  $\overline{Y} = X$ .

**1.1. The Two Conjectures.** Now we state the two conjectures of this article.

CONJECTURE 6 (Densely elliptic rational approximability for euclidean spaces).  
Let  $k \in \mathbb{N}$ . Let

$$\mathbb{R}^k = \{(x_1, \dots, x_k) \mid x_i \in \mathbb{R}\}$$

denote the  $k$ -dimensional euclidean space. Then there exists an elliptic rational set  $X \subset \mathbb{R}^k$ , such that,  $X$  is dense in  $\mathbb{R}^k$ .

CONJECTURE 7 (Densely 2-hyperbolic rational approximability for unit spheres).  
Let  $k \in \mathbb{N}$ . Let

$$S^k = \{(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^k x_i^2 = 1\}$$

denote the  $k$ -dimensional unit sphere. Then there exists a 2-hyperbolic rational set  $X \subset S^k$ , such that,  $X$  is dense in  $S^k$  or equivalently there exists a  $2r$ -hyperbolic rational dense subset in any sphere of rational radius  $r$ .

REMARK 8. Consider the two 1-dimensional real manifolds

$$\mathbb{R}, S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Not only both Conjectures 6, 7 hold true for  $k = 1$ , i.e.  $\mathbb{R}$  is densely elliptic rational approximable and rationally dense because we could take the dense set to be the set of all elliptic rationals.  $S^1$  is densely 2-hyperbolic rational approximable and is rationally dense for we have Conjecture 7 holding true using the following Theorem 9. Also refer to [1] by Paul D. Humke and Lawrence L. Krajewski for a characterization of circles in the plane whose rational points are dense in their respective circles. The conjectures of dense rational approximability of spheres of rational radii and of euclidean spaces of dimension  $k$  are of interest for  $k > 1$  (refer to [2], Chapter 1, Problem 10, Parts 1,2). In this article we prove that the two Conjectures 6, 7 are equivalent in Theorem 17 for all positive integers  $k$ .

### 1.2. Structure of the paper and the main results.

In Section 2, we prove in Theorem 9 that any circle with rational radius is rationally dense.

In Section 3, we prove the main Theorem 15 of this paper, where a homeomorphism is introduced which preserves the rationality property of distances for the dense sets of a certain type, the rationality property of coordinates of the points in the dense sets of the same type and also mention an important application in Corollary 19. Also as a consequence of the main Theorem 15 we prove the equivalence of Conjectures 6, 7 in Theorem 17.

In Section 4 we pose an open Question 1 after proving Theorem 20 concerning dense rational approximability of any ellipse having a certain type of rational point and having an area a rational multiple of  $\pi$ .

## 2. Density of a rational set on any circle with rational radius

THEOREM 9. *Let  $C$  be any circle with rational radius in the Euclidean plane. Then  $C$  is densely hyperbolic rational approximable.*

PROOF. It is enough to prove for the unit circle centered at the origin by using arbitrary translation and rational dilation. First we prove the following claim.

CLAIM 10.

Let

$$\mathbb{Q}_{tan} = \{\theta \in \mathbb{R} \mid \tan(\theta) \text{ is rational or } \theta = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}.$$

$$C_{\mathbb{Q}} = \{\theta \in \mathbb{R} \mid \cos(\theta), \sin(\theta) \in \mathbb{Q}\}.$$

$$\mathbb{Q}_{tan2} = \{\theta \in \mathbb{R} \mid \tan(\theta) = \frac{q}{p}, p, q \in \mathbb{Z}, \gcd(p, q) = 1, p^2 + q^2 \text{ is a square or}$$

$$\theta = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}.$$

$$\mathbb{Q}_{tan4} = \{\theta \in \mathbb{R} \mid \tan(\theta) = \frac{2pq}{p^2 - q^2}, p, q \in \mathbb{Z}, \gcd(p, q) = 1, p^2 + q^2 \text{ is a square}\}.$$

Then

$$(1) \quad \mathbb{Q}_{tan2} = 2\mathbb{Q}_{tan} = C_{\mathbb{Q}} \text{ and } \mathbb{Q}_{tan4} = 2\mathbb{Q}_{tan2} = 4\mathbb{Q}_{tan}.$$

(2)  $\mathbb{Q}_{\tan}, \mathbb{Q}_{\tan 2} = C_{\mathbb{Q}}, \mathbb{Q}_{\tan 4}$  are dense additive subgroups of  $\mathbb{R}$ .

PROOF OF CLAIM. We prove (1) first. We use some elementary geometry. Let  $C$  be a circle with center  $O$  of unit radius. Let  $A, B$  be two points on the circle such that the  $\text{arc}AB$  subtends an angle  $2\theta$  at the center. Extend  $OA$  to meet the circle again at  $P$ . Then the  $\angle APB = \theta$ .

Now we prove  $\mathbb{Q}_{\tan 2} = C_{\mathbb{Q}}$ . Let  $\theta \in C_{\mathbb{Q}}$ ; then  $\cos(\theta) = \frac{r}{s}, \sin(\theta) = \frac{u}{v}$  for some relatively prime integers  $r, s$  and  $u, v$ . So we have  $\frac{r^2 v^2 + u^2 s^2}{s^2 v^2} = 1$  i.e.  $r^2 v^2 + u^2 s^2 = s^2 v^2$ . If  $\cos(\theta) = 0$  then  $\theta \in \mathbb{Q}_{\tan 2}$ . Let  $\cos(\theta) \neq 0$ . Now we observe that  $\tan(\theta)$  is rational and if for some  $q, p$  relatively prime integers  $\frac{q}{p} = \tan(\theta) = \frac{us}{rv}$ . Then there exists an integer  $t$  such that  $us = tq$  and  $rv = tp$  so  $t^2(p^2 + q^2) = r^2 v^2 + u^2 s^2 = s^2 v^2$ . So  $t^2 \mid s^2 v^2 \Rightarrow t \mid sv$  and  $p^2 + q^2 = (\frac{sv}{t})^2$  a perfect square. So  $\theta \in \mathbb{Q}_{\tan 2}$ . The converse is also clear; i.e. if  $\theta \in \mathbb{Q}_{\tan 2}$  then  $\cos(\theta), \sin(\theta)$  are rational.

Now we prove  $\mathbb{Q}_{\tan 2} = 2\mathbb{Q}_{\tan}$ . Let  $\theta \in \mathbb{Q}_{\tan}$  and if  $\tan(\theta)$  is undefined then  $\theta$  is an odd multiple of  $\frac{\pi}{2}$ . So  $2\theta$  is an integer multiple of  $\pi$ . So  $\tan(2\theta) = 0$  and  $2\theta \in \mathbb{Q}_{\tan 2}$ . If  $\tan(\theta) = 0$  then  $\tan(2\theta) = 0$  so  $2\theta \in \mathbb{Q}_{\tan 2}$ . If  $\tan(\theta) = \frac{q}{p}$  with  $\gcd(q, p) = 1$  then  $\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \frac{2pq}{p^2 - q^2}$ . We observe that  $(p^2 - q^2)^2 + 4p^2 q^2 = (p^2 + q^2)^2$  a perfect square. Hence if  $\tan(2\theta) = \frac{u}{v}$  with  $\gcd(u, v) = 1$  then also  $u^2 + v^2$  is a perfect square because there exists an integer  $t$  such that  $2pq = tu, p^2 - q^2 = tv$ . So  $2\theta \in \mathbb{Q}_{\tan 2}$ . Conversely it is also clear that if  $2\theta \in \mathbb{Q}_{\tan 2}$  and  $\theta \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$  then  $\tan(\theta) = \frac{\sin(2\theta)}{1 + \cos(2\theta)}$  is rational. i.e.  $\theta \in \mathbb{Q}_{\tan}$ .

Hence we have  $\mathbb{Q}_{\tan 2} = 2\mathbb{Q}_{\tan} = C_{\mathbb{Q}}$ . Now we prove  $\mathbb{Q}_{\tan 4} = 2\mathbb{Q}_{\tan 2}$ . If  $\theta \in \mathbb{Q}_{\tan 2}, \theta \neq k\frac{\pi}{2}, k \in \mathbb{Z}$  and  $\tan(\theta) = \frac{q}{p}, p^2 + q^2 = \square, \gcd(p, q) = 1$  then  $\tan(2\theta) = \frac{2pq}{p^2 - q^2}$  and  $2\theta \in \mathbb{Q}_{\tan 4}$ . If  $\theta = k\frac{\pi}{2}$  then  $\tan(2\theta) = \frac{0}{1}$  and hence again  $2\theta \in \mathbb{Q}_{\tan 4}$ . Now conversely if  $\tan(2\theta) = \frac{2pq}{p^2 - q^2}, p^2 + q^2 = \square, \gcd(p, q) = 1$  and  $\theta \neq k\frac{\pi}{2}, k \in \mathbb{Z}$  then we have  $\tan(\theta) = \frac{q}{p}$  or  $-\frac{p}{q}$  and so  $\theta \in \mathbb{Q}_{\tan 2}$ . Hence  $\mathbb{Q}_{\tan 4} = 2\mathbb{Q}_{\tan 2}$ .

We observe that  $\tan(0) = 0, \tan(-\theta) = -\tan(\theta), \tan(\theta + \frac{\pi}{2}) = -\cot(\theta)$  and if  $\theta_1 + \theta_2 \neq (2k+1)\frac{\pi}{2}, \theta_i \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}, i = 1, 2$  for some  $k \in \mathbb{Z}$  then  $\tan(\theta_1 + \theta_2) = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1)\tan(\theta_2)}$ . So  $\mathbb{Q}_{\tan}$  is an additive subgroup.

Now we prove (2). First we observe that for every integer  $k \in \mathbb{Z}$  the function

$$\tan_k : (\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2}) \longrightarrow \mathbb{R}, \theta \mapsto \tan(\theta)$$

is a homeomorphism. Hence the set  $\mathbb{Q}_{\tan}$  is dense in  $\mathbb{R}$ . Now we observe that any finite index additive subgroup of a dense additive subgroup of reals is also dense in reals. This completes the proof of the claim.  $\square$

Continuing with the proof of the theorem it is enough to prove for the unit circle  $S^1$  because of rational dilation and arbitrary translation. We consider two points  $P_1, P_2$  on the circle  $S^1$  in the plane such that

$$P_1 = (\frac{a^2 - b^2}{a^2 + b^2}, \frac{2ab}{a^2 + b^2}), P_2 = (\frac{c^2 - d^2}{c^2 + d^2}, \frac{2cd}{c^2 + d^2})$$

for some integers  $a, b, c, d$  such that  $a^2 + b^2, c^2 + d^2$  are nonzero squares. Then we get that  $d(P_1, P_2)^2 = \frac{4(ad - bc)^2}{(a^2 + b^2)(c^2 + d^2)}$  which is a square of a rational and we have  $(a^2 + b^2)(c^2 + d^2) - (ad - bc)^2 = (ac + bd)^2$ . Let

$$X = \{(\frac{a^2 - b^2}{a^2 + b^2}, \frac{2ab}{a^2 + b^2}) \in S^1 \mid a, b \in \mathbb{Z}, a^2 + b^2 = \square \neq 0\}$$

Using the claim we establish that  $S^1$  is 2-hyperbolic rationally approximable and rationally dense using the set  $X \subset S^1$ . Now by arbitrary translation and rational dilation Theorem 9 follows for all circles with rational radii. Note the coordinates after translation need not be rational but the dense set is a rational set, also, the distances are a fixed rational multiple of hyperbolic rationals.

In fact in the above proof if  $(\frac{a^2-b^2}{a^2+b^2}, \frac{2ab}{a^2+b^2})$  is in the dense set then so are the possible four points

$$\left( \pm \frac{a^2 - b^2}{a^2 + b^2}, \pm \frac{2ab}{a^2 + b^2} \right)$$

and if  $ab \neq 0$  then these four points

$$\left( \pm \frac{(\frac{a+b}{2})^2 - (\frac{a-b}{2})^2}{(\frac{a+b}{2})^2 + (\frac{a-b}{2})^2}, \pm \frac{2(\frac{a+b}{2})(\frac{a-b}{2})}{(\frac{a+b}{2})^2 + (\frac{a-b}{2})^2} \right) = \left( \pm \frac{2ab}{a^2 + b^2}, \pm \frac{a^2 - b^2}{a^2 + b^2} \right)$$

do not belong to the dense set as  $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2) \neq \square$ .  $\square$

REMARK 11. If  $p \neq 0 \neq q, p, q \in \mathbb{Z}, \gcd(p, q) = 1, p^2 + q^2 = \square \in \mathbb{Z}$  and if  $\frac{p}{q} = \frac{c}{d}$  for some  $c, d \in \mathbb{Z}$  then  $c^2 + d^2 = \square$ . If  $p \neq 0 \neq q, p, q \in \mathbb{Z}, \gcd(p, q) = 1, p^2 - q^2 = \square \in \mathbb{Z}$  and if  $\frac{p}{q} = \frac{c}{d}$  for some  $c, d \in \mathbb{Z}$  then  $c^2 - d^2 = \square$ .

### 3. On a homeomorphism between deleted point unit sphere and euclidean space of same dimension

In this section we introduce a map between the unit sphere with a point removed and the euclidean space of the same dimension which preserve rationality of distances of a certain type. We begin with a lemma.

LEMMA 12. *Let*

$$\vec{x} = (x_0, x_1, \dots, x_k), \vec{y} = (y_0, y_1, \dots, y_k), \vec{z} = (z_0, z_1, \dots, z_k)$$

*be three points on the unit  $k$ -dimensional sphere  $S^k$ . Let*

$$\begin{aligned} \vec{x} \odot \vec{y} &= \sum_{i=0}^k x_i y_i = \cos(c), \\ \vec{y} \odot \vec{z} &= \sum_{i=0}^k y_i z_i = \cos(a), \\ \vec{z} \odot \vec{x} &= \sum_{i=0}^k z_i x_i = \cos(b). \end{aligned}$$

*Consider the spherical triangle  $xyz$  on  $S^k$ . Let spherical  $\angle yxz = \alpha$ , spherical  $\angle xzy = \gamma$ , spherical  $\angle xyz = \beta$ . Then we have the cosine rule of spherical trigonometry for a suitable choice of angles for  $a, b, c, \alpha, \beta, \gamma$*

$$\begin{aligned} \cos(a) &= \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha) \\ \cos(b) &= \cos(c)\cos(a) + \sin(c)\sin(a)\cos(\beta) \\ \cos(c) &= \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma) \end{aligned} \tag{13}$$

PROOF. It is easy to see that  $\vec{y} - \vec{x} \cos(c)$ ,  $\vec{z} - \vec{x} \cos(b)$  are the tangent vectors at  $x \in S^k$  tangential to the great circles  $C_{xy}$  containing  $x, y$  and  $C_{xz}$  containing  $x, z$  respectively. It is also easy to see that

$$\|\vec{y} - \vec{x} \cos(c)\| = \sin(c), \|\vec{z} - \vec{x} \cos(b)\| = \sin(b).$$

So we have

$$(\vec{y} - \vec{x} \cos(c)) \odot (\vec{z} - \vec{x} \cos(b)) = \sin(b) \sin(c) \cos(\alpha)$$

and hence it follows that

$$\cos(a) = \cos(b) \cos(c) + \sin(b) \sin(c) \cos(\alpha)$$

Hence the other equations (13) also follow similarly.  $\square$

REMARK 14. In the above Lemma 12 a suitable choice of angles for  $a, b, c, \alpha, \beta, \gamma$  exist.

Now we state the main result, Theorem 15 of the paper and this section before giving a corollary.

THEOREM 15. Let  $O = (0, 0, \dots, 0) \in \mathbb{R}^{k+1}$  denote the origin and let  $A = \vec{e} = (1, 0, 0, \dots, 0) \in S^k$ . Let  $X = \vec{x} = (x_0, x_1, \dots, x_k), Y = \vec{y} = (y_0, y_1, \dots, y_k) \in S^k \setminus \{\pm \vec{e}\}$ . Let  $U = \vec{u}, V = \vec{v}$  denote the vectors on the unit sphere corresponding to angular bisectors of  $\angle AOX$  and  $\angle AOY$  in the planes of triangles  $\triangle AOX, \triangle AOY$  respectively. Consider the  $k$ -dimensional tangent space

$$\mathbb{T} = \mathbb{T}_e(S^k) = \{1\} \times \mathbb{R}^k \subset \mathbb{R}^{k+1}$$

tangent to the sphere  $S^k$  at  $A = \vec{e}$ . Suppose after extending the vectors  $U = \vec{u}, V = \vec{v}$ , they meet the tangent space  $\mathbb{T}$  at the points  $B, C$  respectively. Let  $\mathbb{P}$  denote the plane of the triangle  $\triangle ABC$  in the tangent space  $\mathbb{T}$ . If

$$\angle AOX, \angle AOY \in \mathbb{Q}_{\tan 4} = 4\mathbb{Q}_{\tan} \text{ (refer to Claim 10 for definition)}$$

then the following are equivalent.

- (1)  $\|\vec{x} - \vec{y}\|$  is rational.
- (2)  $\sin(\frac{\angle XOY}{2})$  is rational.
- (3) The side length  $BC$  of  $\triangle ABC \subset \mathbb{P}$  is rational.
- (4) The set  $X = \{A, B, C\} \subset \mathbb{P}$  is a rational set.

PROOF. For  $k = 1$  the theorem is trivial. So let us assume  $k > 1$ . First we observe that

$$\|\vec{x} - \vec{y}\|^2 = (\vec{x} - \vec{y}) \odot (\vec{x} - \vec{y}) = 2(1 - \cos(\angle XOY)) = 4\sin^2(\frac{\angle XOY}{2}).$$

Hence (1) and (2) are equivalent.

Now we observe that if  $\tan(\frac{\angle AOX}{2}) = \frac{s}{r}, \tan(\frac{\angle AOY}{2}) = \frac{v}{u}$  such that  $r^2 + s^2, u^2 + v^2$  are squares then by the cosine rule of spherical trigonometry applied to the spherical triangle  $xye$  on the sphere  $S^k$  we have

$$\begin{aligned} & \left(\frac{r^2 - s^2}{r^2 + s^2}\right)\left(\frac{u^2 - v^2}{u^2 + v^2}\right) + \left(\frac{2rs}{r^2 + s^2}\right)\left(\frac{2uv}{u^2 + v^2}\right)\cos(\angle BAC) = \vec{x} \odot \vec{y} \\ \Rightarrow \sin^2(\frac{\angle XOY}{2}) &= \frac{r^2v^2 + s^2u^2 - 2rsuv\cos(\angle BAC)}{(r^2 + s^2)(u^2 + v^2)} = \frac{BC^2}{(OB^2)(OC^2)} \end{aligned}$$

Since  $r^2 + s^2, u^2 + v^2$  are squares the sides  $OB, OC$  are rational. Hence we conclude that (2) and (3) are equivalent.

Now we observe that  $AB^2 = \|\vec{x} - \vec{e}\|^2 = 4\sin^2(\frac{\angle AOX}{2})$  and  $AC^2 = \|\vec{y} - \vec{e}\|^2 = 4\sin^2(\frac{\angle AOY}{2})$  which are both squares of rationals because  $\angle AOX, \angle AOY \in \mathbb{Q}_{tan4}$  by hypothesis. Hence (3) and (4) are equivalent.

This proves the theorem.  $\square$

Now we prove the theorem on equivalence of conjectures in this section. Before we state the theorem we define a homeomorphism from the unit sphere with a point removed and the euclidean space of the same dimension.

DEFINITION 16 (Definition of the homeomorphism). Let  $k > 0$  be a positive integer. Let

$$S^k = \{(x_0, x_1, \dots, x_k) \mid \sum_{i=0}^k x_i^2 = 1\}.$$

Let  $O = (0, 0, \dots, 0), A = \vec{e} = (1, 0, \dots, 0) \in \mathbb{R}^{k+1}$ . Let  $\mathbb{R}^k$  denote the  $k$ -dimensional euclidean space which is identified with

$$\mathbb{T}_e(S^k) = \{1\} \times \mathbb{R}^k \subset \mathbb{R}^{k+1}$$

in a standard way. Then consider the following homeomorphism

$$\phi : S^k \setminus \{-\vec{e}\} \longrightarrow \{1\} \times \mathbb{R}^k$$

defined as follows. Let  $P = \vec{x} = (x_0, x_1, \dots, x_k) \in S^k \setminus \{-\vec{e}\}$ . Consider the unit vector  $U = \vec{u} \in S^k$  which corresponds to the angular bisector of  $\angle AOP$ . After extending this vector suppose it meets the tangent space  $\mathbb{T}_e(S^k) = \{1\} \times \mathbb{R}^k$  at a point  $Q$ . Now define

$$\phi(P) = Q.$$

This is clearly a homeomorphism of the spaces  $S^k \setminus \{-\vec{e}\}, \mathbb{R}^k$ . We note that  $\phi(A) = A$ .

Now we state the theorem.

THEOREM 17. (1) *With the notations in Definition 16 a elliptic dense rational set  $Y \subset \mathbb{R}^k$  also containing the point  $A$  such that the distance from  $A$  are elliptic rationals gets mapped to a dense rational set  $X \subset \mathbb{S}^k$  containing the point  $A = \vec{e}$  such that the distances from  $A = \vec{e}$  are twice of hyperbolic rationals.*

(2) *Conjectures 6, 7 are equivalent.*

PROOF. (1) follows immediately from the previous Theorem 15 because  $\phi$  is a homeomorphism.

(2) follows because we can assume without loss of generality by distance preserving symmetries and rational dilations for euclidean spaces that the dense sets in each of the spaces contain the point  $A$  and from which the distances in the dense set are elliptic rationals in the euclidean spaces and the distances in the dense set are twice of hyperbolic rationals in the unit sphere.

This proves the theorem.  $\square$

Here we prove a corollary of Theorem 15. Before stating a corollary we need a definition.

DEFINITION 18. Let  $k > 0$  be a positive integer. We say a non-empty set  $X \subset \mathbb{R}^k$  is an integral set if the distance set  $\Delta(X) = \{d(x, y) \mid x, y \in \mathbb{R}^k\} \subset \mathbb{N} \cup \{0\}$ .

Now we state the corollary.

**COROLLARY 19.** *Let  $k > 0$  be a positive integer.*

- (1) *There exists an infinite rational set in  $\mathbb{R}^k$  not contained in any hyperplane and also not contained in any  $(k - 1)$ -dimensional sphere in  $\mathbb{R}^k$ .*
- (2) *There exists an infinite rational set in  $\mathbb{S}^k$  not contained in any  $(k - 1)$ -dimensional sphere in  $\mathbb{S}^k$ .*
- (3) *There exists arbitrarily large finite integral  $n$ -set in  $\mathbb{Z}^k \subset \mathbb{R}^k$  with  $n > k$  which is not contained in any hyperplane and also not contained in any  $(k - 1)$ -dimensional sphere in  $\mathbb{R}^k$ .*

**PROOF.** Let  $X \subset S^1 \cap \mathbb{Q}^2 \subset \mathbb{R}^2$  be the dense rational set introduced in Theorem 9. Fix a rational  $r_0 \in \mathbb{Q}^+$ , such that  $1 + r_0^2$  is a square of a rational. There are a few steps involved in the proof, for lifting rational sets to one more higher dimension.

- (1) Dilate the set  $X_1 = X \cup \{(0, 0)\}$  by  $r_0$  to obtain  $r_0 X_1$ .
- (2) Consider  $\{1\} \times r_0 X_1$  a rational set in  $\{1\} \times \mathbb{R}^2 \subset \mathbb{R}^3$ .
- (3) Using the homeomorphism  $\phi$  in Theorem 9 we obtain a rational set  $X_2 \subset S^2$ . If  $Q = (1, r_0 x_1, r_0 x_2) \in \{1\} \times r_0 X_1$  then

$$\tilde{Q} = \frac{1}{\sqrt{1 + r_0^2}}(1, r_0 x_1, r_0 x_2) \in S^2$$

and if

$$\tilde{Q} = (\cos(\alpha), \sin(\alpha)\cos(\beta), \sin(\alpha)\sin(\beta))$$

has rational coordinates then

$$\begin{aligned} P = (\cos(2\alpha) = 2\cos^2(\alpha) - 1, \sin(2\alpha)\cos(\beta) = 2\cos(\alpha)\sin(\alpha)\cos(\beta), \\ \sin(2\alpha)\sin(\beta) = 2\cos(\alpha)\sin(\alpha)\sin(\beta)) \end{aligned}$$

also has rational coordinates and the homeomorphism  $\phi$  in Theorem 9 takes the point  $P$  to the point  $Q$ , i.e.  $\phi(P) = Q$ . So the rational set  $X_2 \subset S^2 \cap \mathbb{Q}^3$ , i.e., has rational coordinates since  $X \subset S^1 \cap \mathbb{Q}^2$  also has rational coordinates.

By induction or repeating this lifting procedure we obtain rational sets in higher dimensional euclidean spaces and also in higher dimensional unit spheres. We observe that the following properties remain invariant during lifting procedure.

- (1) The rational distance property between the points of the lifted sets.
- (2) The rationality property of the coordinates of the lifted sets.
- (3) By including origin the points in the rational sets do not lie on any hyperplane and they do not lie on any sphere.
- (4) Because the coordinates are all rational in the infinite rational sets we can obtain arbitrarily large finite sets with integer coordinates and non-negative integer distances by clearing denominators.

This proves the corollary.  $\square$

#### 4. An open question on densely rational approximability

We have observed that the geometric set which is a line in any dimensional euclidean space is densely pythagorean (elliptic, hyperbolic) rational approximable and the geometric set which is a circle with rational radius in any dimensional euclidean space is densely hyperbolic rational approximable. In order to motivate and



pose the open question we prove a slightly general Theorem 20 concerning hyperbolic rational approximability of ellipses which contain a certain type of rational point and having an area a rational multiple of  $\pi$ .

THEOREM 20. *Let*

$$E = \{(x, y) \in \mathbb{R}^2 \mid ax^2 + by^2 = 1, \text{ where } a, b \text{ are positive rationals.}\}$$

*Then*

(1) *Existence of a rational point of a certain type: If*

$$(x_0, y_0) = \left( \frac{\cos(\theta_0)}{\sqrt{a}}, \frac{\sin(\theta_0)}{\sqrt{b}} \right) \in E(\mathbb{Q})$$

*for some  $\theta_0$  which is not a rational multiple of  $2\pi$  then  $E(\mathbb{Q})$  is dense in  $E(\mathbb{R})$ .*

(2)  *$E(\mathbb{R})$  is rationally dense, i.e. there exists*

$$X \subset E(\mathbb{R}) \cap \mathbb{Q}^2 \text{ such that } \overline{X} = E(\mathbb{R}), \Delta(X) \subset \mathbb{Q}^+ \cup \{0\}.$$

(3) *If in addition  $0 \neq ab = \square \in (\mathbb{Q}^+)^2$  or equivalently the area of the ellipse is a rational multiple of  $\pi$  then  $E(\mathbb{R})$  is densely  $\frac{2}{\sqrt{ab}}$ -hyperbolic rational approximable.*

(4) *Hence any standard ellipse (Eq:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b \in \mathbb{Q}^+$  with its area  $\pi ab$  a rational multiple of  $\pi$ ) is hyperbolic rational approximable.*

PROOF. Let

$$\cos((2k+1)\theta) = \cos(\theta)P_k(\cos^2(\theta)), \sin((2k+1)\theta) = \sin(\theta)Q_k(\sin^2(\theta)),$$

where  $P_k[X], Q_k[X] \in \mathbb{Z}[X], k \in \mathbb{N} \cup \{0\}$ .

Consider the following set  $Y \subset E(\mathbb{Q})$  of points on the ellipse given by

$$Y = \left\{ \left( \frac{\cos((2k+1)\theta_0)}{\sqrt{a}}, \frac{\sin((2k+1)\theta_0)}{\sqrt{b}} \right) \mid k \in \mathbb{Z} \right\} \subset E(\mathbb{Q})$$

We have

$$\begin{aligned} \cos^2(\theta_0) &= ax_0^2, \sin^2\theta_0 = by_0^2, \\ \sin((4k+2)\theta_0) &= 2x_0y_0P_k(\cos^2\theta_0)Q_k(\sin^2\theta_0)\sqrt{ab}, \\ \cos((4k+2)\theta_0) &= ax_0^2P_k^2(\cos^2\theta_0) - by_0^2Q_k^2(\sin^2\theta_0). \end{aligned}$$

Since  $\theta_0$  is not a rational multiple of  $2\pi$  the set  $Y$  is dense in  $E(\mathbb{Q})$ . Also the following set  $X$  is dense. The set  $X$  is given by

$$X = \left\{ \left( \frac{\cos((8k+1)\theta_0)}{\sqrt{a}}, \frac{\sin((8k+1)\theta_0)}{\sqrt{b}} \right) \mid k \in \mathbb{Z} \right\} \subset E(\mathbb{Q})$$

Now consider two points

$$P_1 = \left( \frac{\cos((8k+1)\theta_0)}{\sqrt{a}}, \frac{\sin((8k+1)\theta_0)}{\sqrt{b}} \right), P_2 = \left( \frac{\cos((8l+1)\theta_0)}{\sqrt{a}}, \frac{\sin((8l+1)\theta_0)}{\sqrt{b}} \right)$$

on the ellipse  $E(\mathbb{Q}) \subset E(\mathbb{R})$ . Then we have

$$\begin{aligned}
d(P_1, P_2)^2 &= \frac{4\sin^2(4(k-l)\theta_0)}{ab} \\
&= \frac{4\left(\sin((4k+2)\theta_0) - \sin((4l+2)\theta_0)\right)^2}{ab} \\
&= \frac{4\left(\sin((4k+2)\theta_0)\cos((4l+2)\theta_0) - \cos((4k+2)\theta_0)\sin((4l+2)\theta_0)\right)^2}{ab} \\
&= 16x_0^2y_0^2\left(P_k(\cos^2\theta_0)Q_k(\sin^2\theta_0)(ax_0^2P_l^2(\cos^2\theta_0) - by_0^2Q_l^2(\sin^2\theta_0))\right. \\
&\quad \left.- P_l(\cos^2\theta_0)Q_l(\sin^2\theta_0)(ax_0^2P_k^2(\cos^2\theta_0) - by_0^2Q_k^2(\sin^2\theta_0))\right)^2
\end{aligned}$$

which is a square of a rational. Hence  $E(\mathbb{R})$  is rationally dense and  $\overline{X} = E(\mathbb{R})$ .

If in addition  $ab = \square \in (\mathbb{Q}^+)^2$  then we have

$$d(P_1, P_2)^2 = \frac{4}{ab}\sin^2(4(k-l)\theta_0) = \frac{4}{ab}\frac{u^2}{v^2}$$

where  $v^2 - u^2 = \square \in \mathbb{N}$ . This is because

$$\sin(4(k-l)\theta_0) = \frac{2\tan(2(k-l)\theta_0)}{1 + \tan^2(2(k-l)\theta_0)} = \frac{u}{v}$$

with

$$\begin{aligned}
\tan(2(k-l)\theta_0) &= \frac{\sin(2(k-l)\theta_0)}{\cos(2(k-l)\theta_0)} \\
&= \frac{\sin((2k+1)\theta_0)\cos((2l+1)\theta_0) - \cos((2k+1)\theta_0)\sin((2l+1)\theta_0)}{\cos((2k+1)\theta_0)\cos((2l+1)\theta_0) + \sin((2k+1)\theta_0)\sin((2l+1)\theta_0)}
\end{aligned}$$

is a rational.

Let us consider a standard ellipse  $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ . Then the point

$$(a\cos(\theta_0), b\sin(\theta_0)) = \left(\frac{3a}{5}, \frac{4b}{5}\right) \in E(\mathbb{Q}).$$

Here we observe that  $\theta_0$  is not a rational multiple of  $2\pi$  because  $(\cos(\theta_0) + i\sin(\theta_0))^n = \left(\frac{3+4i}{5}\right)^n \neq 1$  for all  $0 \neq n \in \mathbb{Z}$ . Because the only roots of irreducible cyclotomic polynomials in  $\mathbb{Q}[i]$  are  $\pm 1, \pm i$ . This proves the theorem.  $\square$

Now we pose an interesting question in this article.

**QUESTION 1.** *Let  $k$  be a positive integer. Let  $f \in \mathbb{Z}[X_1, X_2, \dots, X_k]$  be a polynomial with integer coefficients in  $k$  variables. Let*

$$X = \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k \mid f(x_1, x_2, \dots, x_k) = 0\}$$

*be an infinite set. Let  $(X, d) \subset (\mathbb{R}^k, d)$  be the metric space, where  $d$  is the usual euclidean metric. Determine whether  $(X, d)$  is densely rational approximable and the type of rational approximability.*

## 5. Acknowledgments

I would like to thank Prof. R. Balasubramanian, Prof. Vijay Kodiyalam, The Institute of Mathematical Sciences (IMSc.), Chennai, India. The author is supported by an IMSc. scholarship during the period of this work.

## References

- [1] P. D. Humke and L. L. Krajewski. A characterization of circles which contain rational points. *The American Mathematical Monthly*, **86**(4), April 1979, 287–290.
- [2] V. Klee and S. Wagon. Old and new unsolved problems in plane geometry and number theory. *The Dolciani Mathematical Expositions*, No. **11**, Publ. MAA

THE INSTITUTE OF MATHEMATICAL SCIENCES, CHENNAI, TAMILNADU 600113, INDIA  
*E-mail address:* `anilkcp@imsc.res.in`